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Finite-Temperature Black Hole Thermodynamics and Maximal Acceleration

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We show that horizon divergences for scalar fields in infinitely massive black hole backgrounds can be eliminated by resorting to a maximal acceleration principle.

1. INTRODUCTION

The three decades have seen much in black hole physics. One of the results that stimulated researchers is the so-called *Bekenstein–Hawking formula*, which states that a black hole, at least a quasistatic and semiclassical one, behaves like a thermodynamic object whose entropy is proportional to the horizon area [1–3, 13, 14]. Several authors have noted that the appropriate approach to the study of black hole thermodynamics is the statistical one, based on counting the number of particle gravitational states [5, 7, 19, 21, 22]. Within such an approach, however, one finds that the infinite number of states appearing on a black hole horizon introduces entropy divergences of ultraviolet origin. This circumstance led 't Hooft to regularize divergences by means of a model, the so-called *brick wall model*, in which particles are not allowed to be arbitrarily close to the horizon [19]. In this way, the entropy turns out to be proportional to the horizon area, but is divergent as the cutoff distance ϵ approaches zero. In other words, the effect of the parameter ϵ is to cut off modes near the horizon (see also refs. 11 and 20).

An attempt to give to the introduction of a horizon cutoff a meaning other than a technical one (the need to avoid infinities) has been pursuit by McGuigan [15], who argued that the cutoff ϵ can be related to a maximal acceleration which appears in string theory. Actually, the naive observation

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that horizon divergences arise by virtue of the equivalence principle and thermodynamic arguments suggest one to look for a dynamical interpretation of the cutoff ϵ within the pointlike formalism. With this in mind, ref. 21 on infinitely massive black holes looks particularly interesting. Here ϵ appears indeed as a cutoff on the Rindler coordinate, so that it may be intrinsically related to an upper limit on accelerations of pointlike particles.

In this paper we study a massive scalar field in an infinitely massive black hole background when a maximal acceleration principle for pointlike fields is taken into account *ab initio*. To this end, we analyze the Klein–Gordon equation over the modified Rindler space-time suggested in ref. 10. We evaluate the field free energy and elucidate the role played by the maximal acceleration principle in removing the horizon divergences.

The outline of the paper is as follows. In Section 2, we briefly write down essential formulas of Caianiello's model. In Section 3, we consider the Klein–Gordon equation in modified Rindler manifold and discuss its general solutions. In Section 4, we derive the corresponding free energy, making use of a WKB approximation. Section 5 gives conclusions.

2. MAXIMAL ACCELERATION AND CAIANIELLO'S EMBEDDING MODEL

In the context of quantum physics, the conjecture of a maximal acceleration for pointlike particles dates to Caianiello [8, 9]. Since then, the maximum acceleration principle has been discussed in different contexts (see, e.g., refs. 6, 10, and 12 and references therein). It has been in particular realized that the development of the theory of maximal acceleration relies on the differential geometry of the space-time tangent bundle $\mathbb{TM} = \mathbb{M}_4 \otimes \mathbb{TM}_4$, where $\mathbb{M}_4$ is the space-time base manifold and the four-velocity space $\mathbb{TM}_4$ is the fiber manifold. Denoting coordinates of a generic point in the bundle as

$$\{X^A; A = 0, \ldots, 7] = \left\{x^\mu, \frac{c^2}{A_{\max}} v^\mu; \mu = 0, \ldots, 3\right\} \tag{1}$$

where x^μ and v^μ are the usual space-time and four-velocity coordinates, respectively, we find for the line element in $\mathbb{TM}$

$$d\tau^2 = dX^A\, dX_A = dx^\mu\, dx_\mu + \frac{c^4}{A_{\max}^2}\, dv^\mu\, dv_\mu \tag{2}$$

The metric structure (2) automatically implies both a maximal velocity c and a maximal acceleration $A_{\max}$ [6, 8–10]. Interestingly, quantization can be interpreted as curvature of the relativistic eight-dimensional tangent bundle $\mathbb{TM}$. In $\mathbb{TM}$, indeed, the standard operators of the Heisenberg algebra are represented as covariant derivatives and the quantum commutation relations are interpreted as components of the curvature tensor [8, 9].

Since quantum fields are now defined on $\mathbb{TM}$, one gets eight-dimensional generalization of the field equations (see also ref. 6). In general, these equations are nontrivial. In order to investigate correction induced by the maximal acceleration $A_{\max}$ on the fields equations, one needs effective tools. To this end, the simple suggestion of Caianiello turns out to be useful to adopt an embedding procedure which leads to an *effective* four-dimensional space-time metric, $\tilde{g}_{\mu,\nu}$, induced on $\mathbb{M}_4$ by the metric of $\mathbb{TM}$. This induced metric turns out to be defined according to (henceforth we use natural units $\hbar = c = 1$)

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x_\alpha}{\partial \xi^\nu} + \frac{1}{A_{\max}^2}\frac{\partial \dot{x}^\alpha}{\partial \xi^\mu}\frac{\partial \dot{x}_\alpha}{\partial \xi^\nu} \tag{3}$$

where $\dot{x}^\alpha = \dot{x}^\alpha(\xi^\nu)$ is the velocity field obtained by solving the equations of motion in $\mathbb{M}_4$ and expressed in terms of the coordinates ξ^ν chosen to parametrize $\mathbb{M}_4$ [10]. Hence, one gets effective space-time metrics also containing one-loop terms depending on maximal acceleration $A_{\max}$. These terms are expected to play a key role in closing the horizons, where accelerations would exceed $A_{\max}$.

3. MASSIVE SCALAR QUANTUM FIELD THEORY IN MODIFIED RINDLER SPACE-TIME

The geometry outside the horizon of an infinitely massive black hole is described by the Rindler metric, which is a reparametrization of the Minkowskian one. If x^μ, $\mu = 0, \ldots, 3$, denote the Minkowskian space-time coordinates, then Rindler coordinates $(\eta, s, \mathbf{x}_\perp)$ are defined by means of the transformation [18]

$$x_0 = s \sinh \eta, \qquad x_1 = s \cosh \eta, \qquad \mathbf{x}_\perp = (x^2, x^3) \tag{4}$$

where $\eta \in (-\infty, \infty)$ and $s \in [0, \infty)$. In terms of the Rindler coordinates, the Minkowskian line element takes the cylindrical form [we use the metric signature $(-, +, +, +)$]

$$dl^2 = -s^2\, d\eta^2 + ds^2 + d\mathbf{x}_\perp^2 \tag{5}$$

As is well known, the Rindler coordinate system is the coordinate system associated with the Fermi–Walker nonrotating tetrad carried by a uniformly accelerating observer [16]. Lines of constant s and $\mathbf{x}_\perp$ are trajectories of uniformly accelerated observers in the x_1 direction of Minkowski space, s^{-1} being the proper acceleration. Metric (5) thus describes a constant, static, homogeneous gravitational field. The most important property is the occurrence of a horizon located at $s = 0$. By virtue of the equivalence principle, a generic four-dimensional static space-time endowed with a horizon can be regarded, near the horizon, as of the Rindler type.

Let us now consider a massive scalar field ϕ propagating into a Rindler space-time modified according to Caianiello's embedding procedure. Such a space-time has been derived in ref. 10 and is defined by the line element

$$dl^2 = -(s^2 - A_{\max}^{-2})\, d\eta^2 + ds^2 + d\mathbf{x}_\perp^2 \tag{6}$$

In the minimally coupled case, the Klein–Gordon equation on the modified Rindler background (6) reads

$$\left[\frac{-1}{s^2 - A_{\max}^{-2}}\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial s^2} + \frac{s}{s^2 - A_{\max}^{-2}}\frac{\partial}{\partial s} + \sum_{i=2,3}\left(\frac{\partial}{\partial x^i}\right)^2 - M^2\right]\phi(\eta, s) = 0 \tag{7}$$

where M denotes the mass of ϕ. Expanding the field ϕ as

$$\phi = \psi(s)\exp(-i\omega\eta + i\mathbf{k}\cdot\mathbf{x}_\perp)$$

we find that Eq. (7) takes the form

$$\left[\frac{\omega^2}{s^2 - A_{\max}^{-2}} + \frac{\partial^2}{\partial s^2} + \frac{s}{s^2 - A_{\max}^{-2}}\frac{\partial}{\partial s} - \xi^2\right]\psi(s) = 0 \tag{8}$$

where $\xi^2 = M^2 + \mathbf{k}^2$. Setting $u = \xi s$, we then get the modified Bessel equation of order $i\Omega = i(\omega^2 + \xi^2 A_{\max}^{-2})^{1/2}$ with an inhomogeneous term:

$$\left[u^2\frac{\partial^2}{\partial u^2} + u\frac{\partial}{\partial u} - (u^2 - \omega^2 - \xi^2 A_{\max}^{-2})\right]\psi(u) = \xi^2 A_{\max}^{-2}\frac{\partial^2\psi}{\partial u^2} \tag{9}$$

The inhomogeneous solution can be obtained if two linearly independent solutions of the homogeneous equation are known (see, e.g., ref. 17):

$$\psi^{\text{inh}}(u) = \xi^2 A_{\max}^{-2} I_{i\Omega}(u)\int^u K_{i\Omega}(z)\left(z^{-1}\frac{\partial^2\psi^{\text{inh}}(z)}{\partial z^2}\right) - M^{-2}A_{\max}^{-2}K_{i\Omega}(u)\int^u I_{i\Omega}(z)\left(z^{-1}\frac{\partial^2\psi^{\text{inh}}(z)}{\partial z^2}\right) + \text{const} \tag{10}$$

In virtue of the asymptotic behavior at large arguments of the Bessel functions, the K function can be selected as the solution of the homogeneous part. So the solution of Eq. (9) can be written as

$$\psi(s) = C_o K_{i\Omega}(\xi s) + \psi^{\text{inh}}(\xi s) \tag{11}$$

where $\psi^{\text{inh}}(\xi s)$ is determined via the integrodifferential equation (10). As customary, a perturbative approach can be used in the general case [4, 17].

It is worthwhile to note that the presence of the inhomogeneous term in Eq. (9) allows us to regulate the theory by demanding the vanishing of the field ϕ at the horizon without being obliged to introduce an unclear condition for the Rindler frequencies [21]. Such a feature is easily shown also in the special case of a massless scalar field in the bidimensional Rindler space-time. When $\xi = 0$, in fact, the general solution to Eq. (8) is defined in terms of hypergeometric functions as

$$\psi(s) = C_1 F\left(i\omega, -i\omega; \frac{1}{2}; \frac{1}{2}(A_{\text{max}}s + 1)\right) + C_2\sqrt{(A_{\text{max}}s + 1)}F\left(i\omega + 1, -i\omega + 1; \frac{3}{2}; \frac{1}{2}(A_{\text{max}}s + 1)\right) \tag{12}$$

where C_1 and C_2 are arbitrary constants. The request $\lim_{s\to A_{\text{max}}^{-1}} \phi = 0$ is then satisfied for $C_2 = -C_1\, \omega \coth(\pi\omega)$, with C_1 to be determined through the normalization condition.

4. STATISTICAL MECHANICS OF A MASSIVE SCALAR FIELD IN MODIFIED RINDLER SPACE

In this section, we focus on the statistical mechanics of a scalar field in a modified Rindler background (6). After performing the coordinate transformation

$$A_{\text{max}}s = \cosh\rho \quad (\rho \geq 0), \qquad A_{\text{max}}s = -\cosh\rho \quad (\rho \leq 0)$$

we find that the Klein–Gordon equation (8) takes a Schrödinger-like form. This allows us to evaluate the number of modes N_ω, with $\omega \gtrsim M/A_{\text{max}}$, by means of a WKB approximation. We have

$$\pi N_\omega = \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} d\rho\, \sqrt{\omega^2 - (\mathbf{k}^2 + M^2)A_{\text{max}}^{-2}\sinh^2(\rho)}$$

i.e.,

$$N_\omega = \frac{\omega^2 A_{\text{max}}}{4\sqrt{\mathbf{k}^2 + M^2}}\, F\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{\omega^2 A_{\text{max}}^2}{\mathbf{k}^2 + M^2}\right) \tag{13}$$

The free energy is defined via (see, e.g., ref. 21)

$$F = S_h \beta^{-1} \int \frac{d^2 k}{4\pi^2} \int d\omega \, \frac{dN_\omega}{d\omega} f(\omega) \tag{14}$$

where S_h is the area of the horizon, $\beta = 1/k_B T$, $f(\omega) = \ln [1 - \exp(-\omega/k_B T)]$, and $dN_\omega/d\omega$ can be evaluated from (13),

$$\frac{dN_\omega}{d\omega} = \frac{\omega A_{\max}}{2\sqrt{\mathbf{k}^2 + M^2}} \left[F\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{\omega^2 A_{\max}^2}{\mathbf{k}^2 + M^2}\right) - \frac{\omega^2 A_{\max}^2}{8(\mathbf{k}^2 + M^2)} F\left(\frac{3}{2}, \frac{3}{2}; 3; -\frac{\omega^2 A_{\max}^2}{\mathbf{k}^2 + M^2}\right)\right]$$

Equation (14) therefore takes the form

$$F = \frac{S_h}{4\pi\beta} \int d\omega \, f(\omega)[I_1 - I_2] \tag{15}$$

where

$$I_1 = A_{\max}\omega \int dk \, \frac{k}{\sqrt{k^2 + M^2}} F\left(\frac{1}{2}, \frac{1}{2}; 2; -\frac{\omega^2 A_{\max}^2}{\mathbf{k}^2 + M^2}\right) \tag{16}$$

and

$$I_2 = \frac{\omega^3 A_{\max}^3}{8} \int dk \, \frac{k}{(k^2 + M^2)^{3/2}} F\left(\frac{3}{2}, \frac{3}{2}; 3; -\frac{\omega^2 A_{\max}^2}{\mathbf{k}^2 + M^2}\right) \tag{17}$$

Integrating over k in (16)–(17), we get in particular

$$I_1 - I_2 = \omega^2 A_{\max}^2 \left\{ \left[\frac{2\sqrt{2\pi}}{3\Gamma(3/4)^2} + F\left(\frac{1}{2}, \frac{3}{2}; 3; -1\right)\right] - \frac{M}{A_{\max}\omega} F\left(\frac{-1}{2}, \frac{1}{2}; 2; -\frac{\omega^2 A_{\max}^2}{M^2}\right) - \frac{A_{\max}\omega}{8M} F\left(\frac{1}{2}, \frac{3}{2}; 3; -\frac{\omega^2 A_{\max}^2}{M^2}\right)\right\}$$

In the limit $(\omega A_{\max})/M \to \infty$, the function $I_1 - I_2$ tends asymptotically to the value $a \simeq 0.57$, so that formula (15) gives rise to a free energy leading term of the standard form

$$F(\beta) \cong -\frac{\pi^3 a S_h A_{\max}^2}{180\ \beta^4} \tag{18}$$

The result (18) qualitatively agrees with results obtained by other authors [7, 20–22]. The only difference is the numerical coefficient, which, in general, depends on the cutoff procedure.

Before we conclude, a comment is in order. It is instructive, in fact, to consider a rough approximation of Eq. (9). It is realized by neglecting the inhomogeneous term so that a modified Bessel equation of order $i\Omega = i\sqrt{\omega^2 + M^2 A_{\max}^{-2}}$ arises. The free energy in this case can be easily evaluated following the standard procedure. In particular, one obtains

$$F(\beta) \cong \frac{S_h k_B T}{4\pi^2} \int_0^\infty d\omega \ln\left[1 - \exp\left(\frac{-\omega}{k_B T}\right)\right] \times \left[A_{\max}^2 \omega^2 + \frac{\omega}{\Omega} \frac{M^2}{2} \ln\left(\frac{1 - \sqrt{1 - (M/A_{\max}\Omega)^2}}{1 + \sqrt{1 - (M/A_{\max}\Omega)^2}}\right)\right] \tag{19}$$

Compare (19) to the formula obtained by Susskind and Uglum [ref. 21, Eq. (2.20)] and note that we get the same leading term in the high-frequency (low-mass) regime, $F(\beta) \cong -\pi^2 S_h A_{\max}^2 \beta^{-2}/180$, and a slightly different behavior in the low-frequency (large-mass) regime. Thus, neglecting the inhomogeneous term of Eq. (8) furnishes the link to the model of Susskind and Uglum. It is interesting to note that such an approximation can be obtained by shifting the horizon via a conformal transformation of the Rindler metric,

$$G_{\mu\nu}^{\text{Rin}} \to (1 - s^2 A_{\max}^{-2}) G_{\mu\nu}^{\text{Rin}}$$

5. CONCLUSIONS

The origin of divergences of thermodynamic quantities in black hole backgrounds derives from the infinite number of states appearing on the horizon. In order to deal with finite expressions, one thus needs to introduce a cutoff parameter ϵ that shifts the horizon. This parameter can be seen as an upper limit on acceleration for geodesic motion of fields in black hole backgrounds. In this paper, we studied a quantum scalar field in a static space-time with horizon, taking into account a maximal acceleration principle *ab initio*. In particular, we used the modified Rindler space-time suggested in ref. 10, for which, in practice, the horizon is shifted from $s = 0$ to $s = A_{\max}^{-1}$. General solutions to the Klein–Gordon equation were discussed. We also computed the free energy of massive scalar fields in this geometry, obtaining a revised integral formula for the free energy of a scalar field in a Rindler-like background. As expected, it turns out to be finite, and in the limit $(\omega A_{\max})/M \to \infty$, it reduces to the well-known Planckian-like form. The inverse of maximal acceleration plays the role of the cutoff parameter ϵ of the brick-wall-inspired models. The physical origin of horizon divergences thus turns out to be related to the violation of the maximal acceleration principle.

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